THE SHOENFIELD ABSOLUTENESS LEMMA

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ABSTRACT

A new proof of Lévy's version of the Absoluteness Lemma is given—a proof which avoids dependent choices and leads to stronger versions of the lemma.

1. **Introduction.** Shoenfield, in his now classical paper *The Problem of Predicativity* [8], showed in ZF that Σ_2^1 predicates over ω imply their relativizations to L, the constructible universe, and hence that Σ_2^1 subsets of ω are constructible. Lévy used this in his memoir [5] to show in $ZF + DC$, where DC is the axiom of dependent choices, that Σ_1 statements of set theory relativize to L; i.e., that for any Σ_1 sentence ϕ, $\phi \rightarrow \phi^{(L)}$ is a theorem of $ZF + DC$. This has been improved in Jensen-Karp [4] where it is shown (apparently in $ZF + AC$) that if ϕ is Σ_1 and $V_\alpha \models \phi$ then $L_\alpha \models \phi$, provided that α is a limit of admissible ordinals. In this paper we present simple new proofs of these and stronger results.

The proof of the Absoluteness Lemma in the second section is very elementary, and should be intelligible to anyone who understands the proof that L is a model of $ZF + AC$. The proof is carried out in ZF without any form of the axiom of choice.** (This is of interest since the absoluteness lemma is used to show that certain results provable in $ZF + V = L$ are already provable in ZF. See, for example, Theorem 44 of Lévy [5].) Using this and the completeness theorem for the language $L_{\omega_1\omega}$, we give a simple proof of a result of which Shoenfield's original version is a special case.

We turn in Section 3 to refinements along the lines of the Jensen-Karp result mentioned above. The proofs consist in analyzing our arguments of Section 2

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* Research partially supported by N.S.F. Grant GP-8625.

** Kunen first pointed out that Lévy's version is true in every countable model of ZF, using forcing, and hence that it is a theorem of ZF. We are thus exhibiting a proof of Theorem 1a in ZF, a proof which Kunen has shown must exist.

Received April 16, 1970 and in revised form May 17, 1970.

a little more closely to see what closure conditions on L are actually used. In particular, we see that the crucial property of the ordinal α mentioned above is not that it be a limit of admissible ordinals, but rather the weaker "β-property".

2. A simple proof of the absoluteness lemma. We begin by recalling a few well known facts.

Let $\mathscr{L}$ be a countable first order language with equality $=$, a binary relation symbol ε and auxiliary relation symbols. We exclude constant and function symbols since they complicate things. They can be introduced via relation symbols in the usual way so that our results still hold. Structures for the language $\mathscr{L}$ are thus of the form $\mathfrak{A} = \langle A, E, \cdots \rangle$ where $E \subseteq A \times A$ is the interpretation of ε.

Given a sentence ϕ of $\mathscr{L}$ one can effectively find an expansion $\mathscr{L}'$ of $\mathscr{L}$ with new relation symbols $R_1 \cdots R_k$ and an $\forall\exists$-sentence ϕ' of $\mathscr{L}'$ such that:

(i) every model $\mathfrak{A}$ of ϕ can be expanded to a model

$$\langle \mathfrak{A}, R_1, \cdots R_k \rangle \text{ of } \phi';$$

(ii) the reduct of any model of ϕ' to the language $\mathscr{L}$ is a model of ϕ.

This version of the Skolem normal form does not use the axiom of choice. See Church [3], p. 240 for a proof; or, for a more transparent proof in a special case, see Rabin [6], p. 289.

Let ϕ be an $\forall\exists$-sentence, say

$$\forall x_1 \cdots \forall x_n \, \exists y_1 \cdots \exists y_m \, \theta(x_1 \cdots x_n, y_1 \cdots y_m)$$

where θ is quantifier free. Let

$$\mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots \subseteq \mathfrak{B}_k \subseteq \cdots$$

be a chain of structures for $\mathscr{L}$ and let $\mathfrak{B} = \bigcup_{k \in \omega} \mathfrak{B}_k$. If for each k and each $b_1, \cdots, b_n \in B_k$ there is some $l \geqq k$ such that $\mathfrak{B}_l \models \exists y_1 \cdots y_m \theta[b_1 \cdots b_n]$ then $\mathfrak{B} \models \phi$. This remark, together with the Skolem normal form, provides a useful method for constructing models; it is essentially the method used by Gödel in the original proof of the completeness theorem.

A structure $\mathfrak{A} = \langle A, E, \cdots \rangle$ for $\mathscr{L}$ is *well founded* if for every $A_0 \subseteq A$:

(*) if $A_0 \neq 0$ then there is an $a \in A_0$ such that no $b \in A_0$ satisfies bEa.

It is a theorem of ZF that $\mathfrak{A}$ is well founded just in case there is a function $f: A \to$ Ordinals such that bEa implies $f(b) < f(a)$. We say that such an f *demonstrates* that $\mathfrak{A}$ is well founded. In particular, since all theorems of ZF relativize to L we see that *if* $\langle A, E \rangle$ *is constructible, and if* (*) *holds for all con-*

structible $A_0 \subseteq A$ *then* $\mathfrak{A}$ is *well founded*, since there will be a (constructible) function f which demonstrates the well foundedness of $\mathfrak{A}$. It is a theorem of $ZF + AC$ that $\mathfrak{A}$ is well founded just in case there is no sequence $\langle a_n : n \in \omega \rangle$ such that $a_{n+1} E a_n$ for all $n \in \omega$. Again, relativizing to L, where AC holds, and using the contrapositive of the above, we see that *if* $\langle A, E \rangle$ *is constructible, but not well founded, then there is a constructible sequence* $\langle a_n : n \in \omega \rangle$ *such that* $a_{n+1} E a_n$ *for all* $n \in \omega$. This is the key fact behind the Shoenfield Absoluteness Lemma.

Also recall that if $\mathfrak{A}$ is well founded and a model of the axiom of extensionality then there is a unique transitive $(x \in y \in B \Rightarrow x \in B)$ model $\mathfrak{B} = \langle B, \in_B, \cdots \rangle$ which is isomorphic to $\mathfrak{A}$. If $\mathfrak{A}$ is constructible so is $\mathfrak{B}$. We use $r(x)$ for the set theoretic rank of x, i.e. $r(x) = \cup \{ r(y) + 1 : y \in x \}$.

THEOREM 1a. (Shoenfield-Lévy). *Let* ϕ *be a sentence true in a transitive structure* $\mathfrak{A} = \langle A, \in_A, R_1 \cdots R_k \rangle$. *There is a constructible, constructibly countable, transitive model* $\mathfrak{B}$ *of* ϕ.

PROOF. We can assume that the axiom of extensionality is a logical consequence of ϕ. By the Skolem normal form, we can also assume that ϕ is an $\forall\exists$-sentence, say

$$\forall x_1 \cdots \forall x_n \psi(x_1 \cdots x_n)$$

where $\psi(x_1 \cdots x_n)$ is

$$\exists y_1 \cdots \exists y_m \theta(x_1 \cdots x_n, y_1 \cdots y_m)$$

and θ is quantifier free.

Let $\alpha = r(A)$ and let X be the set of pairs $\langle \mathfrak{B}, f \rangle$ such that $\mathfrak{B} = \langle B, E, R_1 \cdots R_k \rangle$ is a finite structure with $B \subseteq \omega$, $f : B \to \alpha$, and f demonstrates that $\mathfrak{B}$ is well founded. Our constructible model of ϕ will be isomorphic to the union $\mathfrak{B} = \bigcup_n \mathfrak{B}_n$ of a constructible chain of structures. The functions f_n must be carried along to insure that $\mathfrak{B}$ is well founded. Thus, we define $\langle \mathfrak{B}_0, f_0 \rangle \subset \langle \mathfrak{B}_1, f_1 \rangle$ if (1)–(3) hold:

(1) $\mathfrak{B}_0 \subseteq \mathfrak{B}_1$ as structures

(2) $f_0 \subseteq f_1$

(3) for any $b_1 \cdots b_n \in B_0$, $\mathfrak{B}_1 \models \psi[b_1 \cdots b_n]$.

We also write $\langle \mathfrak{B}_1, f_1 \rangle \prec \langle \mathfrak{B}_0, f_0 \rangle$ if $\langle \mathfrak{B}_0, f_0 \rangle \subset \langle \mathfrak{B}_1, f_1 \rangle$ (this is not a misprint!).

The pair $\langle X, \prec \rangle$ is a constructible partial ordering, since the members of X are constructible objects, and the definitions of X and $\prec$ are absolute. We will show that it cannot be well founded. Thus, there is a constructible sequence

$$\langle \mathfrak{B}_0, f_0 \rangle \subset \langle \mathfrak{B}_1, f_1 \rangle \subset \cdots \subset \langle \mathfrak{B}_n, f_n \rangle \subset \cdots$$

of elements of X. The union $\mathfrak{B} = \bigcup_n \mathfrak{B}_n$ is constructible, constructibly countable, a model of ϕ by (3), and is well founded since $f = \bigcup_n f_n$ demonstrates its well foundedness. It is thus isomorphic to the desired model. To show that $\langle X, \prec \rangle$ is not well founded we exhibit a non empty subset $X_0 \subset X$ with no minimal element. Let X_0 be the set of those $\langle \mathfrak{B}, f \rangle \in X$ such that there is an embedding i of $\mathfrak{B}$ into $\mathfrak{A}$ satisfying $f(b) = r(i(b))$ for all $b \in B$. The set X_0 is non empty since $\langle \mathfrak{A}_0, \{\langle 0,0 \rangle\} \rangle \in X_0$ where $\mathfrak{A}_0$ is the substructure of $\mathfrak{A}$ with universe $\{0\}$. To show that X_0 has no minimal element, let $\langle \mathfrak{B}_0, f_0 \rangle \in X_0$ with embedding $i_0 : \mathfrak{B}_0 \to \mathfrak{A}$ satisfying $f_0(b) = r(i_0(b))$ for all $b \in B_0$ be given. Let $\mathfrak{A}_0$ be the substructure of $\mathfrak{A}$ isomorphic to $\mathfrak{B}_0$ under i_0. Since

$$\mathfrak{A} \models \forall x_1 \cdots \forall x_n \, \exists y_1 \cdots \exists y_m \, \theta(x_1 \cdots x_n, y_1 \cdots y_m)$$

there is a finite structure $\mathfrak{A}_1$, $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}$, such that for all $a_1 \cdots a_n \in A_0$,

$$\mathfrak{A}_1 \models \exists y_1 \cdots \exists y_n \, \theta[a_1 \cdots a_n].$$

Since $\mathfrak{A}_1$ is finite we can find a $\mathfrak{B}_1$ with $B_1 \subset \omega$, $\mathfrak{B}_0 \subseteq \mathfrak{B}_1$ and an isomorphism $i_1 : \mathfrak{B}_1 \cong \mathfrak{A}_1$ which extends i_0. Define f_1 in the obvious manner:

$$f_1(b) = r(i_1(b))$$

for $b \in B_1$. Then $\langle \mathfrak{B}_1, f_1 \rangle \in X_0$ and $\langle \mathfrak{B}_1, f_1 \rangle \prec \langle \mathfrak{B}_0, f_0 \rangle$.

REMARK. If ϕ is a Σ_1 sentence with only ε as constant, then we can prove $\phi \to \phi^{(L)}$ as follows. Assume ϕ. Then there is a transitive A s.t. $\phi^{(A)}$. Thus there is a constructible transitive B such that $\phi^{(B)}$ and hence $\phi^{(L)}$ since Σ_1 sentences persist upwards and $B \subseteq L$.

We can deduce relativized versions of Theorem 1a by first extending Theorem 1a from a single sentence ϕ to a set T of sentences. Recall that our language $\mathscr{L}$ is countable. We assume that the development of the syntax of $\mathscr{L}$ has been sensibly carried out in ZF so that formulas of $\mathscr{L}$ are elements of HF, the set of hereditarily finite sets, and so that the elementary syntactic notions are Δ_1^{ZF} over HF. In particular we assume that the function of i which tells us the number of places r_i of the symbol R_i is Δ_1^{ZF} over HF. By $L(x)$, for $x \subseteq HF$, we mean the collection of sets constructible from x.

THEOREM 2a. *Let T be a set of sentences true in some transitive structure $\mathfrak{A} = \langle A, \in_A, R_1, R_2, \cdots \rangle$. There is a transitive model $\mathfrak{B} = \langle B, \in_B, R_1', R_2', \cdots \rangle$ of T which is an element of $L(T)$ and which is countable in $L(T)$.*

PROOF. For this proof it is convenient to modify the definition of substructure so that two structures can have a different number of relations. That is, if $\mathfrak{B}_0 = \langle B_0, E_0, R_1 \cdots R_{k_0}\rangle$ and $\mathfrak{B}_1 = \langle B_1, E_1, S_1 \cdots S_{k_1}\rangle$ then we say that $\mathfrak{B}_0 \subseteq \mathfrak{B}_1$, if $k_0 \leqq k_1$ and if

$$\langle B_0, E_0, R_1 \cdots R_{k_0}\rangle \subseteq \langle B_1, E_1, S_1 \cdots S_{k_0}\rangle$$

in the usual sense. As in the proof of Theorem 1a we can assume that all the sentences of T are in $\forall\exists$-form. Let $\phi_1, \phi_2, \cdots, \phi_i \cdots$ be an ω-enumeration in $L(T)$ of the sentences of T, say ϕ_i is

$$\forall x_1 \cdots \forall x_{n_i} \psi_i(x_1 \cdots x_{n_i})$$

where $\psi_i(x_1 \cdots x_{n_i})$ is

$$\exists y_1 \cdots \exists y_{m_i} \theta_i(x_1 \cdots x_{n_i}, y_1 \cdots y_{m_i})$$

and θ_i is quantifier free. Let $no(i)$ be the least k such that all the relation symbols in $\phi_1 \cdots \phi_i$ are among $R_1 \cdots R_k$. We let X be the set of all triples $\langle \mathfrak{B}, i, f\rangle$ where:

i is an integer $\geqq 1$,

$\mathfrak{B} = \langle B, E, R_1 \cdots R_{no(i)}\rangle$ is a finite structure with $B \subseteq \omega$ and R_j is r_j-ary, and $f: B \to r(A)$ demonstrates that $\mathfrak{B}$ is well founded.

We define $\langle \mathfrak{B}_0, i_0, f_0\rangle \subset \langle \mathfrak{B}_1, i_1, f_1\rangle$ by:

(1) $\mathfrak{B}_0 \subseteq \mathfrak{B}_1$

(2) $i_0 < i_1$

(3) $f_0 \subseteq f_1$

(4) for each $i \leqq i_0$ and for all $b_1 \cdots b_{n_i} \in B_0$, $\mathfrak{B}_1 \models \psi_i[b_1 \cdots b_{n_i}]$.

These conditions allow us to take care of our sentences gradually, adding more elements and more relations as our chain grows. With these indications, the reader should be able to finish the proof.

REMARK. Using this theorem one can show in ZF for every Σ_1^{ZF} formula $\phi(v_0)$ that if $x \subseteq \omega$ then $\phi(x) \to \phi(x)^{(L(x))}$. In particular, if $x \in L$ then $\phi(x) \to \phi(x)^{(L)}$ for $x \subseteq \omega$, or, more generally, for $x \in L_\lambda$, where $\lambda = \omega_1^{(L)}$. To show this, one need only introduce constant symbols into $\mathscr{L}$ and put axioms into T describing x.

A Π_1^1 sentence of $\mathscr{L}$ is a second order sentence of the form

$$\forall R_1 \cdots \forall R_n \phi$$

where ϕ is a first order sentence of $\mathscr{L}$ possibly involving relation symbols in addition to $R_1 \cdots R_n$. Similarly, a Σ_2^1 sentence is one of the form

$$\exists R_1 \cdots \exists R_n \forall R_{n+1} \cdots \forall R_{n+m} \phi$$

where ϕ is first order. We use capital Greek letters to range over second order formulas.

The following result, in the crude form stated here, could be derived from Shoenfield's original version of the Absoluteness Lemma. A more refined version appears in §3.

THEOREM 3a. *Let Φ be a Σ_2^1 statement true in some countable structure $\mathfrak{A} = \langle A, E, R_1 \cdots R_k \rangle$. Then Φ is true in some constructible, constructibly countable, model $\mathfrak{B} = \langle B, E, R_1 \cdots R_k \rangle$.*

PROOF. The result for Σ_2^1 clearly follows from the result for Π_1^1. So let $\Phi(R_1)$ be $\forall R_2 \phi(R_1, R_2)$ and let $\mathfrak{A} = \langle A, E, R_1 \rangle$ be a countable transitive model of $\Phi(R_1)$. The case with more relation symbols is the same. We use the completeness theorem* for the infinitary language $L_{\omega_1\omega}$. For each a let c_a be a constant symbol of $L_{\omega_1\omega}$. For any structure $\mathfrak{A}' = \langle A', E', R_1' \rangle$ with A' a countable set, let $\sigma(\mathfrak{A}')$ be the sentence

$$D(\mathfrak{A}') \wedge \forall x \bigvee \{x = c_a;\ a \in A'\}$$

of $L_{\omega_1\omega}$, where $D(\mathfrak{A}')$ is the conjunction of the diagram of $\mathfrak{A}'$. To say that

$$\mathfrak{A} \models \Phi$$

is equivalent to the statement that

$$\sigma(\mathfrak{A}) \rightarrow \phi(R_1, R_2)$$

is logically valid, and hence, by the completeness theorem for $L_{\omega_1\omega}$, to the statement that it is provable. Thus, by hypothesis,

$$\exists \mathfrak{A} \exists P[\mathfrak{A} = \langle A, E, R_1 \rangle \text{ for some countable set } A \wedge$$

$$P \text{ is a proof in } L_{\omega_1\omega} \text{ of} \sigma(\mathfrak{A}) \rightarrow \phi(R_1, R_2)].$$

This is a Σ_1 statement (cf. Barwise [1]) and by the remark following Theorem 1a (i.e., by Lévy's version of Shoenfield's Lemma), it is true in L. Thus there is a constructible, constructibly countable $\mathfrak{B} = \langle B, E, R_1' \rangle$ such that

$$\sigma(\mathfrak{B}) \rightarrow \phi(R_1, R_2)$$

is provable and hence valid. I.e.,

$$\mathfrak{B} \models \forall R_2 \phi(R_1, R_2).$$

* If we wish avoid to the axiom of choice in this proof then we must formulate $\mathscr{L}_{\omega_1\omega}$ in such a way that the set of subformulas of any formula is clearly countable, so that the completeness proof can be carried out.

REMARKS. 1. The definition that we gave for $\mathfrak{A} = \langle A, E, \cdots \rangle$ to be well founded is naturally expressed by a Π_1^1 sentence Φ_0 of $\mathscr{L}$. Thus if $\mathfrak{A}$ is a countable *transitive* model of a Σ_2^1 sentence Φ_1, then we can apply 3a to get a countable *transitive* model $\mathfrak{B}$ of Φ_1 in L.

2. The hypothesis that $\mathfrak{A}$ is countable is essential in Theorem 3a, even if we drop the requirement that $\mathfrak{B}$ be countable. It is easy to find a Π_1^1 statement which is true in $\mathfrak{A} = \langle A, \in_A, U \rangle$, $U \subseteq A$, just in case A is the set of hereditarily countable sets and $U = 0^{\#}$, Solovay's non-constructible set of integers. $0^{\#}$, being Δ_3^1 over ω, is first order over A. (We use AC in this remark).

3. Let ϕ be a sentence involving only ε such that the only transitive model $\langle A, \in_A \rangle$ of ϕ is $\langle HF, \in_{HF} \rangle$. If $\langle HF, \in_{HF}, R \rangle \models \Phi(R)$ for some Σ_2^1 sentence Φ then by applying Theorem 3 to $\Phi(R) \wedge \phi$, we get a constructible R such that $\langle HF, \in_{HF}, R \rangle \models \Phi(R)$. This is Shoenfield's Absoluteness Lemma.

4. Let ϕ be a sentence of $L_{\omega_1 \omega}$ true in some transitive structure $\mathfrak{A}$. We can use the results here to show that ϕ has a transitive model in $L(\phi)$ if $L_{\omega_1 \omega}$ is defined so that $L(\phi) \models$ "ϕ is a sentence of $L_{\omega_1 \omega}$".

THEOREM 4a. *Let T be a set of Σ_2^1 statements true in some countable structure $\mathfrak{A}$. Then there is a model $\mathfrak{B}$ of T such that $\mathfrak{B}$ is in $L(T)$ and is countable in $L(T)$.*

3. **Bounds.** Given a sentence ϕ true in $\mathfrak{A} = \langle A, \in_A, R_1 \cdots R_k \rangle$ where A is transitive, and $r(A) = \alpha$ we want to discuss where in the constructible hierarchy we can find a constructible transitive model $\mathfrak{B}$ of ϕ. That is, we want to find an upper bound on β such that $\mathfrak{B}$ is known to lie in L_β. A convenient framework for this is the Kripke-Platek theory of admissible ordinals and admissible sets, with which we must assume some familiarity. (See, for example, Barwise [1], Barwise-Gandy-Moschovakis [2] or Jensen-Karp [4].)

An admissible set M has the *β-property* if for every pair $\langle A, E \rangle \in M$, if $\langle A, E \rangle$ is not well founded then there is a non-empty $A_0 \in M$ such that $A_0 \subseteq A$ and A_0 violates the well foundedness of $\langle A, E \rangle$; i.e. for every $a \in A_0$ there is a $b \in A_0$ such that $b E a$. An admissible ordinal α has the β-property if L_α, the set of sets constructible before α, has the β-property.

Many admissible sets do not have the β-property. Let M be such a set and let $\mathfrak{A} = \langle A, E \rangle \in M$ be well founded with respect to subsets $A_0 \subseteq A$ in M, but not well founded. Given $a \in A$, let $\mathfrak{A} \upharpoonright a$ be the substructure of $\mathfrak{A}$ whose universe is the set of $b \in A$ such that bEa. Since M is admissible, $\mathfrak{A} \upharpoonright a$ is well founded just in case there is an $f \in M$ which *canonically* demonstrates that $\mathfrak{A} \upharpoonright a$ is well-founded—canonical in the sense that $f(b) = \sup\{f(c) + 1 : cEb\}$ for bEa. Thus, the set

$$A_1 = \{a \in A: \mathfrak{A} \upharpoonright a \text{ is well founded}\}$$

is Σ_1 definable over M and hence $A_0 = A - A_1$ is Π_1 definable over M. A_0 violates the well foundedness of $\mathfrak{A}$, for suppose $a \in A_0$ but there is no $b \in A_0$ such that bEa. Then, for every bEa there is a unique $f \in M$ which canonically demonstrates the well foundedness of $\mathfrak{A} \upharpoonright b$. We can use Σ_1 reflection to piece these functions together, to get a function which canonically demonstrates the well foundedness of $\mathfrak{A} \upharpoonright a$, contradicting $a \in A_0$. The set A_0, while not an element of M, is an element of any admissible set N with $M \in N$ since A_0 is Π_1 definable over M. These r emarks are implicit in Platek's Thesis.

Now suppose that the M of the previous paragraph is $L_\tau(x)$ for some $x \subseteq HF$ and some ordinal τ, where $L_\tau(x)$ is the set of sets constructible from x before τ. Let $\delta > \tau$ be another ordinal such that $L_\delta(x)$ is admissible. Since $A_0 \in L_\delta(x)$ we can use the canonical well ordering of $L_\delta(x)$ to define a sequence $\langle a_n : n \in \omega \rangle \in L_\delta(x)$ such that $a_{n+1} E a_n$ for all $n \in \omega$.

Motivated by the above discussion, we make the following definitions. Given a set x, we define $\kappa(x)$ to be the least ordinal α such that for some admissible set M, $x \in M$ and $\alpha \notin M$. If x is transitive then $L_{\kappa(x)}(x)$ is admissible and $x \in L_{\kappa(x)}(x)$. We define $\kappa^+(x)$ to be the least ordinal α such that for some admissible set M, $x \in M$, $\kappa(x) \in M$ and $\alpha \notin M$. We define $\kappa^*(x)$ to be the least admissible ordinal $\alpha > \kappa(x)$. Thus, $\kappa(x) < \kappa^*(x) \leqq \kappa^+(x)$. If α is an ordinal, then $\kappa(\alpha)$ is the least admissible ordinal $> \alpha$, $\kappa^*(\alpha)$ is the next admissible ordinal after $\kappa(\alpha)$ and $\kappa^+(\alpha) = \kappa^*(\alpha)$. If $X \subseteq \omega$ then we follow the unfortunate recursion theoretic notation and write ω_1^X for $\kappa(X)$. ω_1 is the least non-recursive ordinal.

THEOREM 1b. *Let ϕ be a sentence true in a transitive structure $\mathfrak{A} = \langle A, \in_A, R_1 \cdots R_k \rangle$ and let $\alpha = r(A)$. There is a transitive model $\mathfrak{B}$ of ϕ in $L_{\kappa^+(\alpha)}$. If $\kappa(\alpha)$ has the β-property, then there is such a $\mathfrak{B}$ in $L_{\kappa(\alpha)}$.*

PROOF. Let $\langle X, \prec \rangle$ be as in the proof of Theorem 1a. $\langle X, \prec \rangle$ is an element of $L_{\kappa(\alpha)}$. If $\kappa(\alpha)$ has the β-property, then there will be a sequence

$$\langle \mathfrak{B}_0, f_0 \rangle \subset \cdots \subset \langle \mathfrak{B}_n, f_n \rangle \subset \cdots$$

which, as a sequence, is an element of $L_{\kappa(\alpha)}$. Then the union $\mathfrak{B} = \bigcup_{n \in \omega} \mathfrak{B}_n$ is an element of $L_{\kappa(\alpha)}$ and the transitive structure isomorphic to $\mathfrak{B}$ will also be an element of $L_{\kappa(\alpha)}$. If $\kappa(\alpha)$ does not have the β-property then we must go on to $L_{\kappa^+(\alpha)}$ to find our descending sequence and hence our model.

REMARK. If α is admissible or the limit of admissible ordinals and has the β-property (all limits of admissibles have the β-property, as we saw above) then $V_\alpha \models \phi$ implies $L_\alpha \models \phi$ for all Σ_1 sentences ϕ. There are many admissibles which have the property β without being limits of admissibles, as shown in Platek's Thesis.

We write $\kappa(x,y)$ for $\kappa(\langle x,y\rangle)$ and similarly for κ^* and κ^+.

THEOREM 2b. *Let T be a set of sentences true in some transitive structure $\mathfrak{A} = \langle A, \in_A, R_1, R_2, \cdots\rangle$ and let $\alpha = r(A)$. There is a transitive model $\mathfrak{B}$ of T in $L_{\kappa^+(T,\alpha)}(T)$. If $L_{\kappa(T,\alpha)}(T)$ has the β-property then there is such a $\mathfrak{B}$ in $L_{\kappa(T,\alpha)}(T)$.*

The reader familiar with stable ordinals will realize that we cannot hope for any sort of effective bound in Theorem 3a. However, if we restrict ourselves to Π^1_1 sentences, then we can get a bound.

THEOREM 3b. *Let Φ be a Π^1_1 sentence true in a countable structure $\mathfrak{A}$. There is a model $\mathfrak{B}$ of Φ in $L_{\kappa^*(\mathfrak{A})}$. If $\kappa(\mathfrak{A})$ has the β-property then there is such a $\mathfrak{B}$ in $L_{\kappa(\mathfrak{A})}$.*

PROOF. Suppose that $\mathfrak{A} = \langle A, E, R_1\rangle$, that $\alpha = \kappa(\mathfrak{A})$ and that $\Phi = \Phi(R_1) = \forall R_2 \phi(R_1, R_2)$. Let M be a countable admissible set with $\mathfrak{A} \in M$ and $\alpha \notin M$. Since $\mathfrak{A} \models \forall R_2 \phi(R_1,R_2)$, the infinitary sentence $\sigma(\mathfrak{A}) \to \phi(R_1,R_2)$ of $L_{\omega_1\omega}$ is valid. By the Completeness Theorem of Barwise [1],

$$\langle M, \in_M\rangle \models \exists P[P \text{ is a proof of } \sigma(\mathfrak{A}) \to \phi(R_1,R_2)]$$

and hence the Σ_1 statement

$$\exists \mathfrak{A} \exists P[P \text{ is a proof of } \sigma(\mathfrak{A}) \to \phi(R_1,R_2)],$$

holds in M and thus in $\langle w, \in_w\rangle$ for some transitive $w \in M$ by Σ_1 reflection. Let $\alpha_0 = r(w)$ so that $\alpha_0 < \alpha$ and hence $\kappa(\alpha_0) \leqq \alpha$. By simply choosing a little larger w if necessary, we can assume that $\kappa(\alpha_0) = \alpha$ and hence that $\kappa^+(\alpha_0) = \kappa^*(\alpha_0) = \kappa^*(\mathfrak{A})$. By Theorem 1b there is a transitive $w' \in L_{\kappa^*(\alpha_0)}$ such that the above Σ_1 sentence is true in w' and hence in $L_{\kappa^*(\alpha_0)}$. Thus, there is a transitive structure $\mathfrak{B} \in L_{\kappa^*(\mathfrak{A})}$ and a proof $P \in L_{\kappa^*(\mathfrak{A})}$ such that P is a proof of $\sigma(\mathfrak{B}) \to \phi(R_1,R_2)$. Thus,

$$\mathfrak{B} \models \forall R_2 \phi(R_1,R_2).$$

If $\kappa(\mathfrak{A})$ has the β-property then Theorem 1b tells us that we can choose $w' \in L_{\kappa(\mathfrak{A})}$ and hence $\mathfrak{B} \in L_{\kappa(\mathfrak{A})}$.

To see what this result tells us in a more concrete situation, we consider sets of reals. By a real, we mean a subset X of ω. If P is a Σ_1^1 set of reals then a simple application of Theorem 1b gives an $X \in P \cap L_\alpha$ where α is the second admissible ordinal $> \omega$; i.e., an X which is hyperarithmetic in Kleene's O. This is well known since one can in fact get an X recursive in O.

The best one can do for Π_2^1 or Σ_2^1 sets of reals is $L_{\delta_2^1}$ where δ_2^1 is the least non-Δ_2^1 ordinal. However, we shall show how to get a bound given an $X \in P$.

Given a set P of reals, we let

$$\mu(P) = \text{the least } \alpha \text{ in } \{\omega_1^X : X \in P\}$$

and $\mu^*(P)$ be the least admissible ordinal $> \mu(P)$.

COROLLARY. *Given a non-empty Π_1^1 set P of reals, there is an X in $P \cap L_{\mu^*(P)}$.*

PROOF. Let $X \in P$ be such that $\omega_1^X = \mu(P)$. Let $\Phi(X)$ be the Π_1^1 sentence defining P over $\langle L_\omega, \in \rangle$ so that $\langle L_\omega, \in, X \rangle \models \Phi(X) \wedge \phi$, where ϕ is as in Remark 3 following Theorem 3a ($HF = L_\omega$). Then there is an $X' \in L_{\kappa^*(X)}$ such that $\langle L_\omega, \in, X' \rangle \models \Phi(X)$ so that $X' \in P$. Since $\kappa^*(X) = \mu^*(P)$, this finishes the proof.

One might hope to replace the bound $\mu^*(P)$ by $\mu(P)$ in the corollary. This is not possible, even if P is Π_2^0. For let P be a non-empty Π_2^0 set of reals which has no hyperarithmetic member. By a result of Gandy (cf. Rogers [7], p. 420) it has an element X such that $\omega_1^X = \omega_1$. Thus, $\mu(P) = \omega_1$. Since every $X \in L_{\omega_1}$ is hyperarithmetic, $L_{\mu(P)} \cap P = \varnothing$.

THEOREM 4b. *Let T be a set of Π_1^1 statements true in some countable $\mathfrak{A}$, and let $\alpha = \kappa(T, \mathfrak{A})$. There is a model $\mathfrak{B}$ of T in $L_{\kappa(T,\alpha)}(T)$. If $L_\alpha(T)$ has the β-property then there is such a $\mathfrak{B}$ in $L_\alpha(T)$.*

REMARK. Note that $\kappa(T,\alpha) \leqq \kappa^*(T, \mathfrak{A})$ and, usually, $\kappa(T,\alpha) < \kappa^+(T, \mathfrak{A})$. From this we could prove a relativized version of the above corollary.

FINAL REMARKS. 1) We have not used the axiom of power set except to state the Remark following Theorem 1b.

2) The reader familiar with set primitive recursive (s.p.r.) functions will realize that the bounds in this section could all be improved slightly. For example, in 1b, if $\kappa(\alpha)$ does not have the β-property, it is not necessary to go all the way to $\kappa^+(\alpha)$. One really only needs the least s.p.r. closed ordinal $\delta > \kappa(\alpha)$, since the chain of structures is s.p.r. in α and $\kappa(\alpha)$ and the collapsing of a well founded model of extensionality to a transitive model is an s.p.r. operation.

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